110 Mathcad[1] Worksheets for Nonlinear Programming

Address: Prof. Dr. K. Schittkowski
Department of Mathematics
University of Bayreuth
D - 95440 Bayreuth

Phone: +921 553278 (office)
+921 32887 (home)

Fax: +921 35557

E-mail: klaus.schittkowski@uni-bayreuth.de
Web: http://www.klaus-schittkowski.de
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Abstract

The purpose of the paper is to introduce a set of nonlinear programming test examples in form of Mathcad worksheets. The availability of nonlinear programming test problems is an important assumption to develop and test optimization codes, to learn how optimization routines behave, or to become familiar with implementation and user interface. The problems are taken from the widely used collection of Fortran subroutines of Hock and Schittkowski [33]. The report presents a summary of 110 test examples that have been transferred to Mathcad, a detailed example, and some numerical results. All worksheets can be downloaded from the home page of the author. A particular advantage of executing these problems from Mathcad is the possibility to plot corresponding objective function surfaces and constraint contours very easily.

1 Introduction

A couple of years ago, the author published two test problem collections for testing nonlinear programming codes, see Hock and Schittkowski [33] and Schittkowski [59]. The Fortran source codes of all test problems are available through the link

`http://www.uni-bayreuth.de/departments/math/~kschittkowski/home.htm`

The usage of the subroutines is documented in Schittkowski [61] and some performance results are found in [60].

Moreover, the test problems are widely used and are contained also in other collections, for example in the Cute library of Bongartz et al. [10], available through the URL

`http://www.cse.clrc.ac.uk/activity/cute`

or the test problem collection of Spellucci [70],

`ftp://ftp.mathematik.tu-darmstadt.de/pub/department/software/opti/`

See also the benchmark test page maintained by Mittelmann

`http://plato.la.asu.edu/bench.html`

In addition, AMPL versions of all test problems of the two collections are available through the links

`http://www.sor.princeton.edu/~rvdb/ampl/nlmodels/hs/index.html`

and

`http://www.sor.princeton.edu/~rvdb/ampl/nlmodels/s/index.html`

see also Fourer et al. [28] for more details about AMPL.

We consider the general optimization problem, to minimize an objective function $f(x)$ under nonlinear equality and inequality constraints,

$$x \in I\!\!R^n : \begin{array}{ll} \min f(x) & \\ g_j(x) = 0 \ , & j = 1, \ldots, m_e \ , \\ g_j(x) \geq 0 \ , & j = m_e + 1, \ldots, m \ , \\ x_l \leq x \leq x_u & \end{array} \tag{1}$$

where x is an n-dimensional parameter vector. Objective function and constraints are supposed to be continuously differentiable on the whole $I\!\!R^n$.

The test problems have been used in the past to develop the nonlinear programming code NLPQL [58], a Fortran implementation of a sequential quadratic programming (SQP) algorithm. The design of the numerical algorithm is founded on extensive comparative numerical tests of Schittkowski [57], Schittkowski et al. [67], and Hock, Schittkowski [34]. To complete the numerical tests, an additional random test problem generator was developed for a major comparative study, see [57].

These efforts indicate the importance of a qualified set of test examples for debugging, validation, performance evaluation, and quantitative numerical comparisons with alternative codes. Although not collected in a very systematically way, the test problems represent all numerical difficulties we observe in practice, for example

1. badly scaled objective and constraint functions,
2. badly scaled variables,
3. non-smooth model functions,
4. ill-conditioned optimization problems,
5. non-regular solutions at points where the constraint qualification is not satisfied,
6. different local solutions,
7. infinitely many solutions.

Academic test problems allow either an analytical or a numerical investigation of all interesting properties, with nearly no or only limited efforts. On the other hand, nonlinear programming problems based on a *real-life* background are often too complex to serve as test problems, are often not available, are not programmed in a standard form as required for massive tests, or contain round-off and truncation errors, in particular if secondary iterative numerical algorithms are included to compute function and gradient values.

To give a first visual impression about the distribution of the number of variables n and the number of constraints, m, we present both in Figures 1 and 2. We see, for example, that about 270 of 306 test problems have not more than 10 variables. In a similar way, the distribution of the number of constraints can be interpreted.

The test problems from the first collection of Hock and Schittkowski [33] were transported to Mathcad 11 to test the available optimizer and to get a graphical environment to visualize the optimization problems, i.e., surface plots of objective function and contour lines of constraints. The mcd-files can be downloaded from the home page of the author,

`http://www.uni-bayreuth.de/departments/math/~kschittkowski/home.htm`

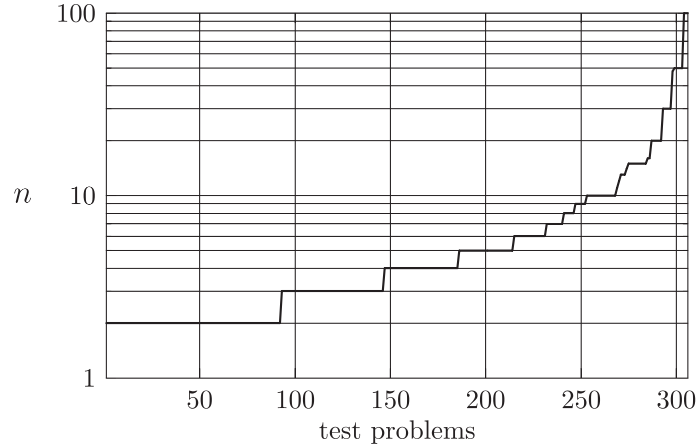

Figure 1: Number of Variables

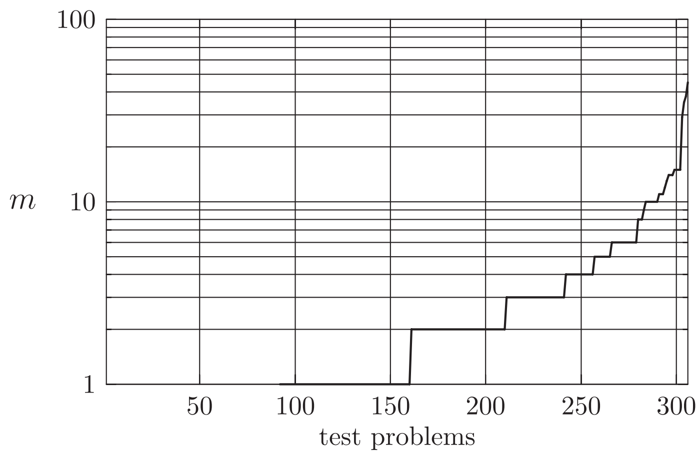

Figure 2: Number of Constraints

The report is one out of a series of Mathcad test problem collections by which numerical routines are tested and the implementation of optimization problems and dynamical systems is outlined, i.e.,

1. data fitting [62],
2. ordinary differential equations [63],
3. differential algebraic equations [64],
4. partial differential equations [65],
5. partial differential algebraic equations [66].

Section 1 contains a list of all nonlinear programming test examples with further details about data, problem structure, and literature. A detailed case study is outlined in Section 2 to illustrate the implementation of a nonlinear optimization problem in Mathcad. An appendix contains a list of all individual results including performance data, number of function calls and number of iterations until successful termination, which have been obtained by the code NLPQLP [60].

2 The Test Problems

A subset of 110 test problems of the collection of Hock and Schittkowski [33] is implemented in form of Mathcad worksheets, i.e., mcd-files. To give at least a first impression about the mathematical structure of a test problem, a classification code is introduced, see Bus [16] or Hock and Schittkowski [33], in the form

OCD-K-s

where the following abbreviations are inserted:

O	Information about objective function
C	Constant objective function
L	Linear objective function
Q	Quadratic objective function
S	Sum of squares
P	Generalized polynomial objective function
G	General objective function

C	Information about constraint functions
U	Unconstrained problem
B	Upper and lower bounds only
L	Linear constraint functions
Q	Quadratic constraint functions
P	Generalized polynomial constraint functions
G	General constraint functions

D	Regularity of the problem
R	Regular problem
I	Irregular problem

K	Information about the solution
T	Exact solution known ('theoretical' problem)
P	Exact solution not known ('practical' problem)

A problem is called a regular one, if the first and second derivatives of all problem functions exist in the feasible region, otherwise an irregular on. **K** = **P** means that the solution of the problem can be obtained only numerically (sometimes called 'real life' problem). The number **s** is replaced by the current serial number within the class of test problems identified by **OCD-K**.

To give an example, consider the problem

$$x_1, x_2 : \begin{array}{l} \min (x_1 - 2)^2 + (x_2 - 1)^2 \\ x_1 + x_2 \le 2 \ \ , \\ x_1^2 - x_2 \le 0 \ \ . \end{array} \tag{2}$$

Since the exact analytical solution is given by $x^\star = (1, 1)$ and since it is the 6-th problem of its class, we classify this problem by **QQR-T-6**.

The subsequently listed optimization problems are provided by the author in form of Mathcad worksheets, and can be downloaded from

`http://www.uni-bayreuth.de/departments/math/~kschittkowski/home.htm`

The following abbreviations are used to characterize test problems for a first quick review:

TP	test problem number
n	number of variables
m_e	number of equality constraints
m	number of constraints
C-NO	classification code
REF	references

TP	n	m_e	m	*C-NO*	*REF*
1	2	0	0	PBR-T-1	Betts [7] (Rosenbrock's banana function)
2	2	0	0	PBR-T-2	Betts [7] (Rosenbrock's banana function)
3	2	0	0	QBR-T-1	Schuldt [68]
4	2	0	0	PBR-T-3	Asaadi [1]
5	2	0	0	GBR-T-1	McCormick [43]
6	2	1	1	QQR-T-1	Betts [7]
7	2	1	1	GPR-T-1	Miele et al. [47]
8	2	2	2	CQR-T-1	Betts [7] (system of equations)
9	2	1	1	GLR-T-1	Miele et al. [47] (trigonometric function)
10	2	0	1	LQR-T-1	Biggs [9]
11	2	0	1	QQR-T-2	Biggs [9]
12	2	0	1	QQR-T-3	Mine at al. [48]
13	2	0	1	QPR-T-1	Betts [7], Kuhn and Tucker [41]
14	2	1	2	QQR-T-4	Bracken and McCormick [13], Himmelblau [31]
15	2	0	2	PQR-T-1	Betts [7] (Rosenbrock's banana function)
16	2	0	2	PQR-T-2	Betts [7] (Rosenbrock's banana function)
17	2	0	2	PQR-T-3	Betts [7] (Rosenbrock's banana function)
18	2	0	2	QQR-T-5	Betts [7]
19	2	0	2	PQR-T-4	Betts [7], Gould [29]
20	2	0	3	PQR-T-5	Betts [7] (Rosenbrock's banana function)
21	2	0	1	QLR-T-1	Betts [7]
22	2	0	2	QQR-T-6	Bracken, McCormick [13], Himmelblau [31], Sheela and Ramamoorthy [69]
23	2	0	5	QQR-T-7	Betts [7]
24	2	0	3	PLR-T-1	Betts [7], Box [12]
25	3	0	0	SBR-T-1	Holzmann [35], Himmelblau [31]
26	3	1	1	PPR-T-1	Huang and Aggerwal [37], Miele et al. [45]
27	3	1	1	PQR-T-6	Miele et al. [46, 47]
28	3	1	1	QLR-T-2	Huang and Aggerwal [37]
29	3	0	1	PQR-T-7	Biggs [9]
30	3	0	1	QQR-T-8	Betts [7]
31	3	0	1	QQR-T-9	Betts [7]
32	3	1	2	QPR-T-2	Evtushenko [26]

(continued)

TP	n	m_e	m	*C-NO*	*REF*
33	3	0	2	PQR-T-8	Beltrami [5], Hartmann [30]
34	3	0	2	LGR-T-1	Eckhardt [24]
35	3	0	1	QLR-T-3	Assadi [1], Charalambous [18], Dimitru and Moga [23], Sheela and Ramamoorthy [69] (Beale's problem)
36	3	0	1	PLR-T-2	Biggs [9]
37	3	0	2	PLR-T-3	Betts [7], Box [12]
38	4	0	0	PBR-T-4	Colville [20], Himmelblau [31] (Colville no. 4)
39	4	2	2	LPR-T-1	Miele et al. [46, 47]
40	4	3	3	PBR-T-2	Beltrami [5], Indusi [38]
41	4	1	1	PLR-T-4	Betts [7], Miele et al. [46]
42	4	2	2	QQR-T-10	Brusch [14]
43	4	0	3	QQR-T-11	Betts [7], Charalambous [18], Sheela and Ramamoorthy [69], Gould [29] (Rosen-Suzuki)
44	4	0	6	QLR-T-4	Konno [40]
45	5	0	0	PBR-T-5	Betts [7], Miele et al. [46]
46	5	2	2	PGR-T-1	Huang and Aggerwal [37], Miele et al. [47]
47	5	3	3	PPR-T-3	Huang and Aggerwal [37], Miele et al. [47]
48	5	2	2	QLR-T-5	Huang and Aggerwal [37], Miele et al. [47]
49	5	2	2	PLR-T-5	Huang and Aggerwal [37]
50	5	3	3	PLR-T-6	Huang and Aggerwal [37]
51	5	3	3	QLR-T-6	Huang and Aggerwal [37]
52	5	3	3	QLR-T-7	Miele et al. [46, 47]
53	5	3	3	QLR-T-8	Betts [7], Miele et al. [46, 47]
54	6	1	1	GLR-T-2	Betts [7], Picket [53]
55	6	6	6	GLR-T-3	Hsia [36]
56	7	4	4	PGR-T-2	Brusch [14]
57	2	0	1	SQR-T-1	Gould [29], Betts [7]
59	2	0	3	GQR-T-1	Himmelblau [31]
60	3	1	1	PPR-T-1	Betts [7], Miele et al. [46, 47]
61	3	2	2	QQR-P-1	Fletcher and Lill [27]
62	3	1	1	GLR-P-1	Betts [7], Picket [53]
63	3	2	2	QQR-P-1	Himmelblau [31], Paviani [51], Sheela and Ramamoorthy [69]
64	3	0	1	PPR-P-1	Best [6]
65	3	0	1	QQR-P-3	Murtagh and Sargent [49]
66	3	0	2	LGR-P-1	Eckhardt [24]
67	3	0	14	GGI-P-1	Colville [20], Himmelblau [31] (Colville no. 8)
68	4	2	2	GGR-P-1	Collani [19] (cost optimal inspection plan)
69	4	2	2	GGR-P-2	Collani [19] (cost optimal inspection plan)
70	4	0	1	SQR-P-1	Himmelblau [31], Himmelblau and Yates [32]
71	4	1	2	PPR-P-3	Bartholomew-Biggs [3]
72	4	0	2	LPR-P-1	Bracken and McCormick [13] (optimal sample size)
73	4	1	3	LGI-P-1	Bracken, McCormick [13], Biggs [9] (cattle-feed)
74	4	3	5	PGR-P-1	Beuneu [8]

(continued)

TP	n	m_e	m	*C-NO*	*REF*
75	4	3	5	PGR-P-2	Beuneu [8]
76	4	0	3	QLR-P-1	Murtagh and Sargent [49]
77	5	2	2	PGR-P-3	Betts [7], Miele et al. [44, 46, 47]
78	5	3	3	PBR-P-4	Asaadi [1], Powell [54]
79	5	3	3	PPR-P-5	Betts [7], Miele et al. [44, 46, 47]
80	5	3	3	GPR-P-1	Powell [54]
81	5	3	3	GPR-P-2	Powell [54]
83	5	0	6	QQR-P-4	Colville [20], Dembo [22], Himmelblau [31] (Colville no. 3)
84	5	0	6	QQR-P-5	Himmelblau [31], Box [11, 12], Betts [7]
85	5	0	38	GGI-P-2	Caroll [17], Himmelblau [31]
86	5	0	10	PLR-P-1	Colville [20], Murtagh and Sargent [49], Himmelblau [31] (Colville no. 1)
87	6	4	4	GGI-P-3	Colville [20], Himmelblau [31] (Colville no. 6)
93	6	0	2	PPR-P-6	Bartholomew-Biggs [3] (transformer design)
95	6	0	4	LQR-P-1	Himmelblau [31], Holzmann [35]
96	6	0	4	LQR-P-2	Himmelblau [31], Holzmann [35]
97	6	0	4	LQR-P-3	Himmelblau [31], Holzmann [35]
98	6	0	4	LQR-P-4	Himmelblau [31], Holzmann [35]
99	7	2	2	GGR-P-3	Betts [7]
100	7	0	4	PPR-P-7	Asaadi [1], Charalambous [18], Wong [72]
101	7	0	6	PPR-P-8	Beck and Ecker [4], Dembo [22]
102	7	0	6	PPR-P-9	Beck and Ecker [4], Dembo [22]
103	7	0	6	PPR-P-10	Beck and Ecker [4], Dembo [22]
104	8	0	6	PPR-P-11	Dembo [22], Rijckaert [55] (optimal reactor design)
105	8	0	1	GLR-P-2	Bracken and McCormick [13] (maximum-likelihood estimation)
106	8	0	6	LQR-P-5	Avriel and Williams [2], Dembo [22] (heat exchanger design)
107	9	6	6	PGR-P-4	Bartholomew-Biggs [3] (static power scheduling)
108	9	0	13	QQR-P-6	Himmelblau [31], Pearson [52]
109	9	6	10	PGR-P-5	Beuneu [8]
110	10	0	0	GBR-P-1	Himmelblau [31], Paviani [51]
111	10	3	3	GGR-P-4	Himmelblau [31], Bracken and McCormick [13], White et al. [71]
112	10	3	3	GLR-P-3	Himmelblau [31], Bracken and McCormick [13], White et al. [71] (chemical equilibrium)
113	10	0	8	QQR-P-7	Asaadi [1], Charalambous [18], Wong [72]
114	10	3	11	QGR-P-6	Bracken and McCormick [13] (alkylation process)
116	13	0	15	LQR-P-6	Dembo [21, 22] (3-stage membrane separation)
117	15	0	5	PQR-P-1	Colville [20], Himmelblau [31] (Colville no. 2, Shell dual)
118	15	0	29	QLR-P-2	Bartholomew-Biggs [3]
119	16	8	8	PLR-P-2	Colville [20], Himmelblau [31] (Colville no. 7)

3 A Mathcad Worksheet Example

Mathcad (`http://www.mathcad.com`) is an interactive GUI with a large number of built-in mathematical functions. Special commands allow to solve also constrained nonlinear programming problems. The subsequent lines describe the usage of these functions and are taken from the Mathcad documentation:

Minimizing or Maximizing a Function

Minimize *(f, var1, var2, ...)* Returns the values of *var1, var2, ...* which satisfy the constraints in a solve block and which make the function *f* take on its smallest value.

Maximize *(f, var1, var2, ...)* Returns the values of *var1, var2, ...* which satisfy the constraints in a solve block and which make the function *f* take on its largest value.

Arguments:

- *var1, var2, ...* are scalar variables found in the solve block. They are defined above the solve block as guess values.
- *f* is a function defined above the solve block. For example, an argument *g* could refer to the function *g(x,y):=x/y*.

Using the functions: To use the *minimize* or *maximize* function:

- Define the function to maximize or minimize.
- Define guess values for the variables being solved for.
- Type the word *Given* to start the solve block.
- Beneath the *Given*, type equalities and inequalities which act as constraints using boolean operators.
- Enter the *Minimize* or *Maximize* function with the appropriate arguments.

Notes:

- These functions return a scalar when only one variable is involved. Otherwise they return a vector whose first element is *var1*, second element is *var2*, and so on.
- If there are no constraints, the word *Given* is not necessary.
- You can type the *Minimize* and *Maximize* functions are not case-sensitive.

To give an impression how a test problem is implemented, we consider problem TP22, where a quadratic objective function is to be minimized subject to one linear and one parabolic inequality constraint,

$$x_1, x_2 : \begin{array}{l} \min(x_1 - 2)^2 + (x_2 - 1)^2 \\ x_1 + x_2 \leq 2 \ , \\ x_1^2 - x_2 \leq 0 \ , \end{array} \tag{3}$$

see also (2). Starting values are $x_0 = (2, 2)$ and optimal solution is $x^\star = (1, 1)$, where both constraints are active. Surface plot of objective function and contour plots of constraints are shown in Figure 3. The diamond shows the position of the optimal solution, the black rectangle the starting point.

Subsequently, the Mathcad implementation is listed, see Figure 4. The mcd-files contain not only data and functions which describe the optimization problem, but also the exact or best known solution, respectively, and the final deviation of the achieved solution from the known one.

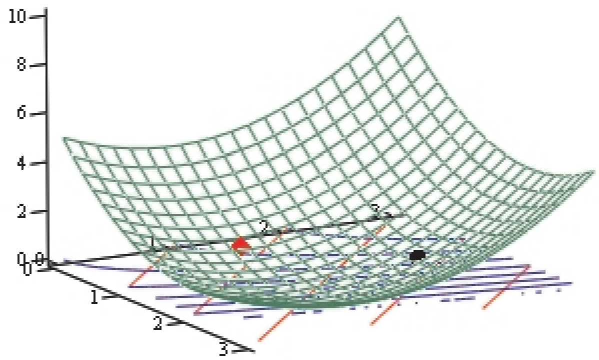

Figure 3: Mathcad Plot

TP22

Nonlinear Programming

Description: Quadratic objective function and constraints (Bracken, McCormick, 1968)

Objective Function: $f(x,y) := (x-2)^2 + (y-1)^2$

Constraint Functions: $g1(x,y) := -x - y + 2$

$g2(x,y) := -x^2 + y$

Starting Values: $x := 2 \qquad y := 2$

Solve: Given $\quad g1(x,y) \geq 0 \quad g2(x,y) \geq 0$

$$\begin{pmatrix} x_{min} \\ y_{min} \end{pmatrix} := \text{Minimize}(f, x, y)$$

Computed Solution: $x_{min} = 1 \qquad y_{min} = 1 \qquad f(x_{min}, y_{min}) = 1$

Optimal Solution: $x_{opt} := 1 \qquad y_{opt} := 1 \qquad f(x_{opt}, y_{opt}) = 1$

Accuracy: $\max(|x_{min} - x_{opt}|, |y_{min} - y_{opt}|) = 4.327 \times 10^{-8}$

$f(x_{min}, y_{min}) - f(x_{opt}, y_{opt}) = 8.654 \times 10^{-8}$

$|\min(0, g1(x_{min}, y_{min}))| + |\min(0, g2(x_{min}, y_{min}))| = 0$

Plot: $n := 20 \qquad j := 0..n \qquad i := 0..n$

$X_{i,j} := -2 + 5 \cdot \frac{i}{n} \qquad Y_{i,j} := -2 + 6 \cdot \frac{j}{n} \qquad Z_{i,j} := f(X_{i,j}, Y_{i,j})$

$U_{i,j} := g1(X_{i,j}, Y_{i,j}) \qquad V_{i,j} := g2(X_{i,j}, Y_{i,j})$

Figure 4: Mathcad Declarations

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APPENDIX: Individual Results for NLPQLP

To show how efficiently these problems can be solved, we present some numerical performance data, i.e., number of function calls, number of iterations, and final termination accuracy. With the default tolerances given, all problems can be solved successfully by the code NLPQLP, a new version of the SQP implementation NLPQL of the author [60]. Derivatives are computed by a five-point-difference formula and termination tolerance is set to 10^{-7}. More details about the test environment and evaluation of successful returns are found in [60, 61].

The subsequent table contains a list of all test problems with the data

- *TP* test problem number,
- *NF* number of objective function evaluations,
- *NDF* number of gradient evaluations of objective function,
- *FEX* exact objective function value,
- *F* computed objective function value,
- *DFX* relative error in objective function,
- *DGX* sum of constraint violations including bound violations.

TP	*NF*	*NDF*	*FEX*	*F*	*DFX*	*DGX*
1	26	19	0.00000000E+00	0.73114619E-10	0.73E-10	0.00E+00
2	20	15	0.50426188E-01	0.50426193E-01	0.11E-06	0.00E+00
3	10	10	0.00000000E+00	0.16103740E-19	0.16E-19	0.00E+00
4	2	2	0.26666667E+01	0.26666667E+01	0.00E+00	0.00E+00
5	8	6	-0.19132230E+01	-0.19132230E+01	0.11E-10	0.00E+00

(continued)

TP	NF	NDF	FEX	F	DFX	DGX
6	10	9	0.00000000E+00	0.19130495E-12	0.19E-12	0.22E-04
7	11	10	-0.17320508E+01	-0.17320508E+01	-0.18E-08	0.11E-07
8	5	5	-0.10000000E+01	-0.10000000E+01	0.00E+00	0.53E-04
9	6	6	-0.50000000E+00	-0.50000000E+00	0.53E-09	0.25E-09
10	27	13	-0.10000000E+01	-0.10000000E+01	0.61E-10	0.82E-14
11	10	9	-0.84984642E+01	-0.84984642E+01	-0.30E-12	0.85E-12
12	9	8	-0.30000000E+02	-0.30000000E+02	-0.58E-09	0.35E-07
13	23	23	0.10000000E+01	0.10002673E+01	0.27E-03	0.00E+00
14	6	6	0.13934650E+01	0.13934650E+01	-0.10E-11	0.77E-12
15	3	3	0.30650001E+01	0.30650000E+01	-0.20E-07	0.37E-09
16	12	8	0.25000000E+00	0.39820605E+01	0.15E+02	0.00E+00
17	19	17	0.99999998E-02	0.99999998E-02	0.72E-11	0.00E+00
18	8	8	0.50000000E+01	0.50000000E+01	-0.11E-08	0.39E-07
19	29	19	-0.69618137E+00	-0.69618137E+00	-0.15E-08	0.79E-07
20	5	5	0.38198730E+02	0.38198730E+02	0.19E-15	0.00E+00
21	5	5	-0.99959998E+00	-0.99959998E+00	-0.89E-11	0.00E+00
22	7	6	0.10000000E+01	0.10000000E+01	-0.22E-11	0.32E-11
23	7	7	0.20000000E+01	0.20000000E+01	0.27E-13	0.00E+00
24	5	5	-0.10000000E+01	-0.10000000E+01	-0.33E-08	0.67E-08
25	23	19	0.00000000E+00	0.57804601E-02	0.58E-02	0.00E+00
26	20	18	0.00000000E+00	0.61563845E-07	0.62E-07	0.59E-04
27	37	22	0.40000000E+01	0.40000000E+01	0.44E-10	0.23E-11
28	5	4	0.00000000E+00	0.24873839E-14	0.25E-14	0.21E-07
29	13	12	-0.22627417E+02	-0.22627417E+02	-0.21E-09	0.69E-08
30	19	17	0.10000000E+01	0.10000000E+01	0.37E-07	0.00E+00
31	12	7	0.60000000E+01	0.60000000E+01	-0.51E-08	0.80E-08
32	3	3	0.10000000E+01	0.10000000E+01	-0.16E-09	0.78E-10
33	5	5	-0.45857864E+01	-0.40000000E+01	0.13E+00	0.00E+00
34	8	8	-0.83403245E+00	-0.83403245E+00	-0.12E-09	0.11E-08
35	7	7	0.11111111E+00	0.11111111E+00	0.15E-12	0.49E-12
36	7	4	-0.33000000E+04	-0.33000000E+04	-0.87E-12	0.26E-10
37	11	10	-0.34560000E+04	-0.34560000E+04	0.16E-13	0.00E+00
38	27	27	0.00000000E+00	0.29634809E-07	0.30E-07	0.00E+00
39	14	12	-0.10000000E+01	-0.10000000E+01	-0.46E-08	0.28E-08
40	6	6	-0.25000000E+00	-0.25000000E+00	-0.93E-09	0.43E-09
41	7	7	0.19259259E+01	0.19259259E+01	0.30E-08	0.26E-10
42	10	8	0.13857864E+02	0.13857864E+02	-0.31E-08	0.17E-07
43	36	11	-0.44000000E+02	-0.44000000E+02	-0.53E-08	0.83E-07
44	6	6	-0.15000000E+02	-0.15000000E+02	0.20E-09	0.00E+00
45	8	8	0.10000000E+01	0.10000000E+01	0.00E+00	0.00E+00
46	14	12	0.00000000E+00	0.55479250E-06	0.55E-06	0.19E-06
47	17	13	0.00000000E+00	0.46052734E-09	0.46E-09	0.93E-07
48	9	8	0.00000000E+00	0.27709246E-07	0.28E-07	0.12E-10
49	34	34	0.00000000E+00	0.71968270E-07	0.72E-07	0.68E-10
50	18	14	0.00000000E+00	0.31208038E-08	0.31E-08	0.91E-12
51	5	3	0.00000000E+00	0.41972609E-18	0.42E-18	0.92E-09
52	8	6	0.53266476E+01	0.53266476E+01	0.39E-10	0.26E-13
53	8	7	0.40930233E+01	0.40930233E+01	0.74E-08	0.41E-11
54	2	2	-0.90807476E+00	-0.72240097E-33	0.10E+01	0.57E-05
55	31	17	0.63333333E+01	0.67733692E+01	0.69E-01	0.71E-14
56	11	9	-0.34560000E+01	-0.34560000E+01	-0.10E-07	0.24E-07
57	13	11	0.28459670E+01	0.28459670E+01	0.16E-09	0.00E+00

(continued)

TP	NF	NDF	FEX	F	DFX	DGX
59	17	15	-0.78042263E+01	-0.67545660E+01	0.13E+00	0.00E+00
60	11	10	0.32568200E-01	0.32568200E-01	0.23E-08	0.67E-08
61	8	7	-0.14364614E+03	-0.14364614E+03	-0.11E-09	0.18E-07
62	14	9	-0.26272514E+05	-0.26272514E+05	-0.47E-12	0.11E-12
63	8	8	0.96171517E+03	0.96171517E+03	0.30E-11	0.14E-09
64	152	84	0.62998424E+04	0.62998424E+04	-0.48E-10	0.70E-11
65	8	8	0.95352886E+00	0.95352882E+00	-0.42E-07	0.54E-06
66	7	7	0.51816327E+00	0.51816327E+00	-0.22E-08	0.31E-08
67	20	20	-0.11620365E+04	-0.11620365E+04	-0.15E-07	0.00E+00
68	40	26	-0.92042502E+00	-0.92042504E+00	-0.18E-07	0.11E-06
69	63	40	-0.95671289E+03	-0.95671289E+03	0.43E-09	0.18E-10
70	37	34	0.74984636E-02	0.74984649E-02	0.17E-06	0.00E+00
71	5	5	0.17014017E+02	0.17014017E+02	-0.26E-08	0.22E-08
72	22	22	0.72767938E+03	0.72767936E+03	-0.25E-07	0.11E-11
73	5	5	0.29894378E+02	0.29894378E+02	0.62E-10	0.17E-11
74	11	11	0.51264981E+04	0.51264981E+04	0.48E-10	0.94E-09
75	9	9	0.51744129E+04	0.51744127E+04	-0.37E-07	0.25E-11
76	6	6	-0.46818182E+01	-0.46818182E+01	0.45E-10	0.00E+00
77	16	15	0.24150513E+00	0.24150513E+00	-0.14E-07	0.34E-07
78	8	8	-0.29197004E+01	-0.29197004E+01	0.49E-10	0.19E-11
79	10	9	0.78776821E-01	0.78776822E-01	0.10E-07	0.30E-07
80	7	7	0.53949848E-01	0.53949847E-01	-0.73E-08	0.72E-08
81	8	8	0.53949848E-01	0.53949846E-01	-0.28E-07	0.27E-07
83	18	7	-0.30665539E+05	-0.30665539E+05	-0.27E-11	0.14E-13
84	67	35	-0.52803351E+02	-0.52361456E+02	0.84E-02	0.21E-04
85	91	56	-0.19051338E+01	-0.19051553E+01	-0.11E-04	0.15E-06
86	6	5	-0.32348679E+02	-0.32348679E+02	0.18E-09	0.14E-10
87	20	16	0.89275977E+04	0.89275977E+04	0.63E-10	0.84E-09
93	15	12	0.13507596E+03	0.13507596E+03	0.12E-07	0.40E-09
95	2	2	0.15619514E-01	0.15619530E-01	0.98E-06	0.00E+00
96	2	2	0.15619513E-01	0.15619530E-01	0.10E-05	0.00E+00
97	7	7	0.31358091E+01	0.31358089E+01	-0.63E-07	0.00E+00
98	7	7	0.31358091E+01	0.31358089E+01	-0.63E-07	0.00E+00
99	52	32	-0.83107989E+09	-0.83107989E+09	0.71E-11	0.45E-09
100	20	14	0.68063006E+03	0.68063006E+03	0.11E-09	0.24E-07
101	68	41	0.18097648E+04	0.18097648E+04	0.61E-11	0.13E-12
102	54	38	0.91188057E+03	0.91188057E+03	0.10E-09	0.75E-13
103	45	31	0.54366796E+03	0.54366796E+03	0.78E-10	0.23E-13
104	16	16	0.39511634E+01	0.39511634E+01	0.44E-09	0.57E-08
105	56	48	0.11384162E+04	0.11384185E+04	0.20E-05	0.00E+00
106	37	37	0.70493309E+04	0.70492480E+04	-0.12E-04	0.19E-11
107	8	8	0.50550118E+04	0.50550118E+04	0.30E-10	0.48E-13
108	14	13	-0.86602540E+00	-0.86602544E+00	-0.37E-07	0.72E-07
109	46	31	0.53620693E+04	0.53620692E+04	-0.18E-07	0.58E-11
110	10	7	-0.45778470E+02	-0.45778470E+02	0.28E-09	0.00E+00
111	51	51	-0.47761090E+02	-0.47761091E+02	-0.13E-07	0.18E-09
112	23	20	-0.47761086E+00	-0.47761074E+00	0.24E-06	0.56E-10
113	16	13	0.24306209E+02	0.24306209E+02	0.16E-09	0.24E-08
114	59	58	-0.17688070E+02	-0.17688070E+02	-0.45E-09	0.32E-06
116	104	70	0.97588409E+02	0.97587510E+02	-0.92E-05	0.40E-07
117	16	16	0.32348679E+02	0.32348679E+02	-0.32E-10	0.00E+00
118	19	19	0.66482045E+03	0.66482045E+03	0.57E-11	0.15E-07

(continued)

TP	NF	NDF	FEX	F	DFX	DGX
119	13	10	0.24489970E+03	0.24489970E+03	0.75E-11	0.64E-12